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Abstract

We present new fast *discrete Helmholtz-Hodge decomposition (DHHD)* methods to efficiently compute at the order $\mathcal{O}(\varepsilon)$ the divergence-free (solenoidal) or curl-free (irrotational) components and their associated potentials of a given $\mathbf{L}^2(\Omega)$ vector field in a bounded domain. The solution algorithms solve suitable penalized boundary-value elliptic problems involving either the $\mathbf{grad}\,(\mathrm{div}\,)$ operator in the *vector penalty-projection (VPP)* or the $\mathbf{rot}\,(\mathbf{rot}\,)$ operator in the *rotational penalty-projection (RPP)* with *adapted right-hand sides* of the same form. Therefore, they are extremely well-conditioned, fast and cheap avoiding to solve the usual Poisson problems for the scalar or vector potentials. Indeed, each (VPP) or (RPP) problem only requires two conjugate-gradient iterations whatever the mesh size, when the penalty parameter ε is sufficiently small. We state optimal error estimates vanishing as $\mathcal{O}(\varepsilon)$ with a penalty parameter ε as small as desired up to machine precision, e.g. $\varepsilon = 10^{-14}$. Some numerical results confirm the efficiency of the proposed (DHHD) methods, very useful to solve problems in electromagnetism or fluid dynamics.

Keywords: Helmholtz-Hodge decompositions, Rotational penalty-projection, Vector penalty-projection, Penalty method, Error analysis, PDE's with adapted right-hand sides.
2010 MSC: 35J20, 35J25, 35Q35, 35Q60, 49M25, 49M30, 65N08, 65N15, 76A02, 76M12, 78A02, 78M12

1. Introduction

Notations. We use below the usual functionnal setting for the Navier-Stokes [25, 17, 12] or Maxwell equations [10]. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3 in practice) be an open bounded and connected domain with a Lipschitz continuous boundary $\Gamma = \partial\Omega$ and $\mathbf{n}$ be the outward unit normal vector on Γ. We assume that either Γ is of class $C^{1,1}$ or Ω is a convex domain. To simplify the presentation in this Note by avoiding the technical construction of vector potentials with cuts inside the domain, we assume that Ω is simply-connected with a connected boundary Γ. Some results can be generalized for a multiply-connected domain Ω; see [7] and also [16, 14, 17] or [1, 2] for theoretical arguments.

We use bold capital letters to denote spaces of vector-valued functions and $(.,.)_0$ for the $L^2(\Omega)$ inner product, $\|.\|_0$ for the $L^2(\Omega)$-norm, $\|.\|_s$ for the Sobolev $H^s(\Omega)$-norm and $\langle .,.\rangle_\Gamma$ for the duality pairing between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. We define below some Hilbert spaces with their usual respective inner product and associated norm:

$$\mathbf{H}_{div}(\Omega) = \left\{\mathbf{u} \in L^2(\Omega)^d;\ \mathrm{div}\,\mathbf{u} \in L^2(\Omega)\right\}, \quad \mathbf{H}_{0,div}(\Omega) = \{\mathbf{u} \in \mathbf{H}_{div}(\Omega),\ \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0 \text{ on } \Gamma\}$$
$$\mathbf{H}_{rot}(\Omega) = \left\{\mathbf{u} \in L^2(\Omega)^d;\ \mathbf{rot}\,\mathbf{u} \in L^2(\Omega)^d\right\}, \quad \mathbf{H}_{0,rot}(\Omega) = \{\mathbf{u} \in \mathbf{H}_{rot}(\Omega),\ \mathbf{u}\wedge\mathbf{n}_{|\Gamma} = 0 \text{ on } \Gamma\}$$
$$\mathbf{H}_{div,rot0}(\Omega) = \{\mathbf{u} \in \mathbf{H}_{div}(\Omega);\ \mathbf{rot}\,\mathbf{u} = 0,\ \mathbf{u}\wedge\mathbf{n}_{|\Gamma} = 0 \text{ on } \Gamma\}$$
$$\mathbf{H}_{rot,div0}(\Omega) = \{\mathbf{u} \in \mathbf{H}_{rot}(\Omega);\ \mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0 \text{ on } \Gamma\}$$
$$\mathbf{H} = \left\{\mathbf{u} \in L^2(\Omega)^d;\ \mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0 \text{ on } \Gamma\right\}, \quad L^2_0(\Omega) = \left\{q \in L^2(\Omega);\ \int_\Omega q\,dx = 0\right\}$$
$$\mathbf{H}^1_{\mathbf{n}}(\Omega) = \left\{\mathbf{u} \in H^1(\Omega)^d;\ \mathbf{u}\cdot\mathbf{n}_{|\Gamma} = 0\right\}, \quad \mathbf{H}^1_{\boldsymbol{\tau}}(\Omega) = \left\{\mathbf{u} \in H^1(\Omega)^d;\ \mathbf{u}\wedge\mathbf{n}_{|\Gamma} = 0\right\}.$$

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We recall the Helmholtz-Hodge orthogonal decomposition of $L^2(\Omega)^d$ for a bounded domain [22, 20] and [25, Theorem 1.5]: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{G}_0 \oplus \mathbf{G}_h$ with $\mathbf{G} = \mathbf{G}_0 \oplus \mathbf{G}_h$ defined as:

$$\mathbf{G} = \mathbf{H}^\perp = \{\mathbf{u} \in L^2(\Omega)^d;\ \mathbf{u} = \mathbf{grad}\,\phi,\ \phi \in H^1(\Omega)/\mathbb{R}\},$$
$$\mathbf{G}_0 \quad = \quad \{\mathbf{u} \in L^2(\Omega)^d;\ \mathbf{u} = \mathbf{grad}\,\phi,\ \phi \in H^1_0(\Omega)\}, \quad \mathbf{G}_h = \mathbf{G}_0^\perp = \{\mathbf{u} \in L^2(\Omega)^d;\ \mathbf{u} = \mathbf{grad}\,\phi,\ \phi \in H^1(\Omega),\ \Delta\phi = 0\}.$$

Thus, for all vector field $\mathbf{v} \in L^2(\Omega)^d$, there exists a unique $(\mathbf{v}_0, \mathbf{v}_h, \mathbf{v}_\psi) \in \mathbf{G}_0 \times \mathbf{G}_h \times \mathbf{H}$ such that:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_h + \mathbf{v}_\psi \quad \text{with} \quad \mathbf{v}_0 = \mathbf{grad}\,\phi_0, \quad \mathbf{v}_h = \mathbf{grad}\,\phi_h \quad \text{and} \quad \mathbf{v}_\psi = \mathbf{rot}\,\boldsymbol{\psi}, \quad \operatorname{div}\boldsymbol{\psi} = 0 \quad \text{in} \ \Omega. \tag{1}$$

Then, $\mathbf{v}_\phi = \mathbf{v}_0 + \mathbf{v}_h = \mathbf{grad}\,\phi \in \mathbf{G}$ and $\mathbf{v}_\psi = \mathbf{rot}\,\boldsymbol{\psi} \in \mathbf{H}$ respectively denote the curl-free (irrotational) and divergence-free (solenoidal) components of $\mathbf{v}$, $\mathbf{v}_h$ having both a null curl and divergence, and $\phi = (\phi_0 + \phi_h) \in H^1(\Omega)/\mathbb{R}$ denotes the scalar potential and $\boldsymbol{\psi} \in \mathbf{H}^1(\Omega)$ the vector potential (for $d = 3$) or scalar stream-function ($d = 2$); see also [17, Theorem 3.6 - Corollary 3.4] and [1, Theorem 3.17]. This gives the following bounds with Pythagore and the mean Poincaré inequality since $\int_\Omega \phi\, dx = 0$:

$$\|\mathbf{v}_\psi\|_0^2 + \|\mathbf{grad}\,\phi\|_0^2 = \|\mathbf{v}\|_0^2 \quad \text{and} \quad \|\phi\|_0 \le c_0(\Omega)\,\|\mathbf{grad}\,\phi\|_0 \le c_0(\Omega)\,\|\mathbf{v}\|_0. \tag{2}$$

If $\mathbf{v}$ belongs to $\mathbf{H}_{div}(\Omega)$ which gives a sense to the normal trace $\mathbf{v}\cdot\mathbf{n}_{|\Gamma}$ in $H^{-\frac{1}{2}}(\Gamma)$, then ϕ (up to an additive constant) and ϕ_0 are the respective solutions in $H^1(\Omega)$ of the following Poisson problems:

$$\Delta\phi = \operatorname{div}\mathbf{v} \quad \textit{in}\ \Omega \quad \text{with} \quad \mathbf{grad}\,\phi\cdot\mathbf{n}_{|\Gamma} = \mathbf{v}\cdot\mathbf{n} \quad \textit{on}\ \Gamma, \quad \text{since} \quad \int_\Omega \operatorname{div}\mathbf{v}\,dx = \langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_\Gamma$$
$$\Delta\phi_0 = \operatorname{div}\mathbf{v} \quad \textit{in}\ \Omega \quad \text{with} \quad \phi_{0|\Gamma} = 0 \quad \textit{on}\ \Gamma.$$

In the sequel, we design new discrete Helmholtz-Hodge decompositions (DHHD) within two or three components which completely get rid of the solution of the Poisson problems for the scalar or vector potentials; see [21, 11, 24, 17, 12, 2, 15, 26, 18]. These decompositions carry out the solution of penalized boundary-value elliptic problems involving either the $\mathbf{grad}\,(\operatorname{div})$ or $\mathbf{rot}\,(\mathbf{rot})$ operators with *adapted right-hand sides* of the same form. Hence, the solution algorithms are extremely well-conditioned, fast and cheap. Typically two iterations of a preconditioned conjugate gradient, whatever the mesh step, are necessary to get the machine precision when the penalty parameter ε is taken sufficiently small as shown in [5, Theorem 1.1 - Corollary 1.3]. These decompositions can be used as fundamental ingredients of efficient methods to solve problems in fluid dynamics or electromagnetism where the vector field solutions must satisfy constraints such that prescribed divergence or curl, see e.g. [4, 6, 8].

2. Approximation of the divergence-free component $\mathbf{v}_\psi = \mathbf{rot}\,\boldsymbol{\psi}$ with (RPP)

We propose below the so-called *rotational penalty-projection*, associated with a non natural normal boundary condition $\mathbf{v}_\psi\cdot\mathbf{n}_{|\Gamma} = (\mathbf{rot}\,\boldsymbol{\psi})\cdot\mathbf{n}_{|\Gamma} = 0$, to directly calculate an accurate and divergence-free approximation $\mathbf{v}^\varepsilon_\psi = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon$ of the solenoidal component $\mathbf{v}_\psi = \mathbf{rot}\,\boldsymbol{\psi}$ of $\mathbf{v}$. The method performs an approximate curl-free projection by enforcing the constraint $\mathbf{rot}\,\mathbf{v}_\psi = \mathbf{rot}\,\mathbf{v}$, *i.e.* $\mathbf{rot}\,(\mathbf{v} - \mathbf{v}_\psi) = 0$ with a penalty method [13]. Thus, for any $\mathbf{v}$ given in $\mathbf{H}_{rot}(\Omega)$, we consider the weak *rotational penalty-projection (RPP)* problem below for all $\varepsilon > 0$:

$$\varepsilon\left(\mathbf{v}^\varepsilon_\psi, \boldsymbol{\varphi}\right)_0 + \left(\mathbf{rot}\,\mathbf{v}^\varepsilon_\psi, \mathbf{rot}\,\boldsymbol{\varphi}\right)_0 = (\mathbf{rot}\,\mathbf{v}, \mathbf{rot}\,\boldsymbol{\varphi})_0\,, \quad \text{for all}\ \boldsymbol{\varphi} \in \mathbf{H}_{rot,div0}(\Omega). \tag{3}$$

In fact, this method is designed to be a suitable approximate method to find, at the limit process when $\varepsilon \to 0$, the unique solution $\mathbf{v}_\psi$ in $\mathbf{H}_{rot}(\Omega)$ of the exact orthogonal curl-projection problem of $\mathbf{v}$ onto $\mathbf{H}$:

$$\mathbf{rot}\,\mathbf{v}_\psi = \mathbf{rot}\,\mathbf{v} \quad \text{and} \quad \operatorname{div}\mathbf{v}_\psi = 0 \ \text{in} \ \Omega \quad \text{with} \quad \mathbf{v}_\psi\cdot\mathbf{n}_{|\Gamma} = 0 \ \text{on} \ \Gamma. \tag{4}$$

The problem (3) is well-posed in $\mathbf{H}_{rot,div0}(\Omega)$ as stated in Theorem 2.1 below; see proof in [7].

Theorem 2.1 (Analysis of the weak rotational penalty-projection (3).). *For all $\varepsilon > 0$ and any $\mathbf{v} \in \mathbf{H}_{rot}(\Omega)$, there exists a unique solution $\mathbf{v}_\psi^\varepsilon$ in $\mathbf{H}_{rot,div0}(\Omega)$ to the weak rotational penalty-projection (3) and $\mathbf{v}_\psi^\varepsilon = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon$ belongs to the space $\mathbf{H}^1(\Omega) \cap \mathbf{H}$ for all $\varepsilon > 0$.*

Moreover, we have the following error estimates:

$$\|\mathbf{v}_\psi - \mathbf{v}_\psi^\varepsilon\|_1 + \|\mathbf{rot}\,(\mathbf{v} - \mathbf{v}_\psi^\varepsilon)\|_0 \leq c(\Omega)\,\|\mathbf{v}\|_0\,\varepsilon, \quad \text{for all } \varepsilon > 0. \tag{5}$$

For all $\varepsilon > 0$ and any $\mathbf{v}$, we consider the strong *rotational penalty-projection (RPP)* problem below for which (3) may be the weak form:

$$(RPP_n) \quad \left\{ \begin{array}{l} \varepsilon\,\mathbf{v}_\psi^\varepsilon + \mathbf{rot}\,\left(\mathbf{rot}\,\mathbf{v}_\psi^\varepsilon\right) = \mathbf{rot}\,(\mathbf{rot}\,\mathbf{v}) \;\text{ in }\; \Omega \qquad \text{with } \left(\mathbf{rot}\,(\mathbf{v}_\psi^\varepsilon - \mathbf{v})\right) \wedge \mathbf{n}_{|\Gamma} = 0, \;\; \mathbf{v}_\psi^\varepsilon \cdot \mathbf{n}_{|\Gamma} = 0 \;\text{ on }\; \Gamma \\ \Rightarrow \quad \mathbf{v}_\psi^\varepsilon = \dfrac{1}{\varepsilon}\mathbf{rot}\,\left(\mathbf{rot}\,(\mathbf{v} - \mathbf{v}_\psi^\varepsilon)\right) = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon, \quad \mathrm{div}\,\mathbf{v}_\psi^\varepsilon = 0, \; \boldsymbol{\psi}^\varepsilon = \dfrac{1}{\varepsilon}\mathbf{rot}\,(\mathbf{v} - \mathbf{v}_\psi^\varepsilon), \; \mathrm{div}\,\boldsymbol{\psi}^\varepsilon = 0 \;\text{ in }\; \Omega. \end{array} \right. \tag{6}$$

We notice that any solution $\mathbf{v}_\psi^\varepsilon$ to (6) writes exactly as a curl, and thus necessarily verifies $\mathrm{div}\,\mathbf{v}_\psi^\varepsilon = 0$.

Proposition 2.2 (Strong solution to (RPP$_n$) problem.). *For $\mathbf{v} \in \mathbf{H}^2(\Omega)$, if we assume that the weak solution $\mathbf{v}_\psi^\varepsilon$ to (3) also belongs to $\mathbf{H}^2(\Omega)$, then $\mathbf{v}_\psi^\varepsilon$ is the strong solution to the problem (6). Moreover, we can choose $\boldsymbol{\psi}^\varepsilon \in \mathbf{H}^1(\Omega)$ such that: $\mathbf{rot}\,(\mathbf{v} - \mathbf{v}_\psi^\varepsilon) = \varepsilon\,\boldsymbol{\psi}^\varepsilon$ and we have $\boldsymbol{\psi}^\varepsilon$ in $\left\{\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\tau}}(\Omega);\ div\,\mathbf{u} = 0\right\}$ satisfying the Gauge condition $div\,\boldsymbol{\psi}^\varepsilon = 0$ with $\boldsymbol{\psi}^\varepsilon \wedge \mathbf{n}_{|\Gamma} = 0$ and the error estimate: $\|\boldsymbol{\psi} - \boldsymbol{\psi}^\varepsilon\|_1 \leq c(\Omega)\,\|\mathbf{v}\|_0\,\varepsilon$ for all $\varepsilon > 0$.*

Besides, the adapted boundary condition $(\mathbf{rot}\,\mathbf{v}_\psi^\varepsilon) \wedge \mathbf{n}_{|\Gamma} = (\mathbf{rot}\,\mathbf{v}) \wedge \mathbf{n}_{|\Gamma}$ on Γ in (6) holds in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Indeed, by imposing this adapted boundary condition on Γ, we really take advantage of the adapted right-hand side *which allows us to include the desired normal condition $\mathbf{v}_\psi^\varepsilon \cdot \mathbf{n}_{|\Gamma} = 0$ in the functional space.*

Remark 1 (Approximation of both potentials $\boldsymbol{\psi}$ and ϕ.). *The (RPP) method yields approximations of order $O(\varepsilon)$ of $\mathbf{v}_\psi$ and $\boldsymbol{\psi}$ with $\mathbf{v}_\psi^\varepsilon = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon$ in $\mathbf{H}$. However, the curl of $(\mathbf{v} - \mathbf{v}_\psi^\varepsilon)$ is only as $O(\varepsilon)$, which prevents us from writing it exactly as a gradient and thus from directly computing an approximation of the scalar potential ϕ. This can be performed with the (VPP) method presented in Section 3.2. Then, the calculation of both the approximate potentials $\boldsymbol{\psi}^\varepsilon$ and ϕ^ε of a DHHD requires the solution of the (RPP) and (VPP) problems to get respectively the couples $(\mathbf{v}_\psi^\varepsilon = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon,\ \boldsymbol{\psi}^\varepsilon)$ and $(\mathbf{v}_\phi^\varepsilon = \mathbf{grad}\,\phi^\varepsilon,\ \phi^\varepsilon)$.*

3. Approximation of the curl-free components $\mathbf{v}_\phi = \mathbf{grad}\,\phi$ and $\mathbf{v}_0 = \mathbf{grad}\,\phi_0$ with (VPP)

3.1. Vector Penalty-Projection (VPP$_\tau$) for $\mathbf{v}_0 = \mathbf{grad}\,\phi_0$

Here, the key idea is to introduce the so-called *the vector penalty-projection*, associated with a non natural tangential boundary condition $\mathbf{v}_0 \wedge \mathbf{n}_{|\Gamma} = (\mathbf{grad}\,\phi_0) \wedge \mathbf{n}_{|\Gamma} = 0$, to directly calculate an accurate and curl-free approximation $\mathbf{v}_0^\varepsilon = \mathbf{grad}\,\phi_0^\varepsilon$ of the irrotational component $\mathbf{v}_0 = \mathbf{grad}\,\phi_0$ of $\mathbf{v}$. The method performs an approximate divergence-free projection by enforcing the constraint $\mathrm{div}\,\mathbf{v}_0 = \mathrm{div}\,\mathbf{v}$, *i.e.* $\mathrm{div}\,(\mathbf{v} - \mathbf{v}_0) = 0$ with a penalty method. Thus, for any $\mathbf{v}$ given in $\mathbf{H}_{div}(\Omega)$, we consider the weak *vector penalty-projection (VPP)* problem below for all $\varepsilon > 0$:

$$\varepsilon\,\left(\mathbf{v}_0^\varepsilon, \boldsymbol{\varphi}\right)_0 + \left(\mathrm{div}\,\mathbf{v}_0^\varepsilon, \mathrm{div}\,\boldsymbol{\varphi}\right)_0 = (\mathrm{div}\,\mathbf{v}, \mathrm{div}\,\boldsymbol{\varphi})_0\,, \quad \text{for all } \boldsymbol{\varphi} \in \mathbf{H}_{div,rot0}(\Omega). \tag{7}$$

In fact, this method is designed to give a suitable approximate sequence to find, at the limit process when $\varepsilon \to 0$, the unique solution $\mathbf{v}_0$ in $\mathbf{H}_{div}(\Omega)$ of the exact orthogonal projection problem of $\mathbf{v}$ onto $\mathbf{G}_0$:

$$\mathrm{div}\,\mathbf{v}_0 = \mathrm{div}\,\mathbf{v} \quad \text{and} \quad \mathbf{rot}\,\mathbf{v}_0 = 0 \;\text{ in }\; \Omega \quad \text{with} \quad \mathbf{v}_0 \wedge \mathbf{n}_{|\Gamma} = 0 \;\text{ on }\; \Gamma. \tag{8}$$

The problem (7) is well-posed in $\mathbf{H}_{div,rot0}(\Omega)$ as stated in Theorem 3.1; see proof in [7].

Theorem 3.1 (Analysis of the weak vector penalty-projection (7).). *For any $\mathbf{v} \in \mathbf{H}_{div}(\Omega)$ and all $\varepsilon > 0$, there exists a unique solution $\mathbf{v}_0^\varepsilon$ in $\mathbf{H}_{div,rot0}(\Omega)$ to the weak vector penalty-projection (7) and $\mathbf{v}_0^\varepsilon = \mathbf{grad}\,\phi_0^\varepsilon$ belongs to the space $\left\{\mathbf{u} \in \mathbf{H}^1_{\boldsymbol{\tau}}(\Omega);\ \mathbf{rot}\,\mathbf{u} = 0,\ \mathbf{u} \wedge \mathbf{n}_{|\Gamma} = 0\right\} \subset \mathbf{G}_0$ for all $\varepsilon > 0$.*

Moreover, we have the following error estimates:

$$\|\mathbf{v}_0 - \mathbf{v}_0^\varepsilon\|_1 + \|div\,(\mathbf{v} - \mathbf{v}_0^\varepsilon)\|_0 \leq c(\Omega)\,\|\mathbf{v}\|_0\,\varepsilon, \quad \text{for all } \varepsilon > 0. \tag{9}$$

If $div\,\mathbf{v} = 0$ with $\int_\Gamma \mathbf{v} \cdot \mathbf{n}\,ds = 0$, then $\mathbf{v}_0^\varepsilon = 0$ and $\phi_0^\varepsilon = 0$ for all $\varepsilon > 0$.

For all $\varepsilon > 0$ and any $\mathbf{v}$, we consider the strong *vector penalty-projection (VPP)* problem below for which (7) may be the weak form:

$$(VPP_\tau) \quad \left\{ \begin{array}{l} \varepsilon\,\mathbf{v}_0^\varepsilon - \mathbf{grad}\left(\mathrm{div}\,\mathbf{v}_0^\varepsilon\right) = -\mathbf{grad}\,(\mathrm{div}\,\mathbf{v}) \ \text{ in } \Omega \qquad \text{with } \ \mathrm{div}\,(\mathbf{v}_0^\varepsilon - \mathbf{v})_{|\Gamma} = 0, \quad \mathbf{v}_0^\varepsilon \wedge \mathbf{n}_{|\Gamma} = 0 \ \text{ on } \Gamma \\ \quad \Rightarrow \quad \mathbf{v}_0^\varepsilon = \dfrac{1}{\varepsilon}\mathbf{grad}\left(\mathrm{div}\,(\mathbf{v}_0^\varepsilon - \mathbf{v})\right) = \mathbf{grad}\,\phi_0^\varepsilon, \quad \mathbf{rot}\,\mathbf{v}_0^\varepsilon = 0,\ \phi_0^\varepsilon = \dfrac{1}{\varepsilon}\mathrm{div}\,(\mathbf{v}_0^\varepsilon - \mathbf{v}) \ \text{ in } \Omega. \end{array} \right. \tag{10}$$

We notice that any solution $\mathbf{v}_0^\varepsilon$ to (10) writes exactly as a gradient, and thus necessarily verifies $\mathbf{rot}\,\mathbf{v}_0^\varepsilon = 0$.

Proposition 3.2 (Strong solution to (VPP$_\tau$) problem.)**.** *For $\mathbf{v} \in \mathbf{H}^2(\Omega)$, if we assume that the weak solution $\mathbf{v}_0^\varepsilon$ to (7) also belongs to $\mathbf{H}^2(\Omega)$, then $\mathbf{v}_0^\varepsilon$ is the strong solution to the problem (10). Moreover, we can choose ϕ_0^ε such that $div\,(\mathbf{v}_0^\varepsilon - \mathbf{v}) = \varepsilon\,\phi_0^\varepsilon$ which gives ϕ_0^ε in $H_0^1(\Omega)$ and the error estimate: $\|\phi_0 - \phi_0^\varepsilon\|_2 \le c(\Omega)\,\|\mathbf{v}\|_0\,\varepsilon$.*

Besides, the adapted boundary condition $(div\,\mathbf{v}_0^\varepsilon)_{|\Gamma} = (div\,\mathbf{v})_{|\Gamma}$ on Γ in (10) holds in $H^{\frac{1}{2}}(\Gamma)$. Indeed, by imposing this adapted boundary condition on Γ, we really take advantage of the adapted right-hand side *which enables us to include the desired tangential condition $\mathbf{v}_0^\varepsilon \wedge \mathbf{n}_{|\Gamma} = 0$ in the functional space.*

3.2. *Vector Penalty-Projection (VPP$_n$) for $\mathbf{v}_\phi = \mathbf{grad}\,\phi$*

For what follows in this Section, the hypothesis Ω simply-connected is not necessary.
The key idea of the *vector penalty-projection method* amounts to directly calculate an accurate and curl-free approximation $\mathbf{v}_\phi^\varepsilon = \mathbf{grad}\,\phi^\varepsilon$ of the irrotational component $\mathbf{v}_\phi = \mathbf{grad}\,\phi$ of $\mathbf{v}$. The method performs an approximate divergence-free projection by enforcing the constraint $\mathrm{div}\,\mathbf{v}_\phi = \mathrm{div}\,\mathbf{v}$, *i.e.* $\mathrm{div}\,(\mathbf{v} - \mathbf{v}_\phi) = 0$ with a penalty method. Here, we actually enforce the divergence condition using the efficient splitting proposed in [5] which yields an *adapted right-hand side* of the same form of the limit *left-hand side operator*. This produces an extremely well-conditioned, fast and cheap method. Thus, for any $\mathbf{v}$ given in $\mathbf{H}_{div}(\Omega)$, we consider the so-called *vector penalty-projection (VPP)* problem for all $\varepsilon > 0$:

$$(VPP_n) \quad \left\{ \begin{array}{l} \varepsilon\,\mathbf{v}_\phi^\varepsilon - \mathbf{grad}\left(\mathrm{div}\,\mathbf{v}_\phi^\varepsilon\right) = -\mathbf{grad}\,(\mathrm{div}\,\mathbf{v}) \ \text{ in } \Omega \qquad \text{with } \ \mathbf{v}_\phi^\varepsilon \cdot \mathbf{n}_{|\Gamma} = \mathbf{v}\cdot\mathbf{n} \ \text{ on } \Gamma, \quad \forall \varepsilon > 0 \\ \quad \Rightarrow \quad \mathbf{v}_\phi^\varepsilon = \dfrac{1}{\varepsilon}\mathbf{grad}\left(\mathrm{div}\,(\mathbf{v}_\phi^\varepsilon - \mathbf{v})\right) = \mathbf{grad}\,\phi^\varepsilon, \quad \mathbf{rot}\,\mathbf{v}_\phi^\varepsilon = 0,\ \phi^\varepsilon = \dfrac{1}{\varepsilon}\mathrm{div}\,(\mathbf{v}_\phi^\varepsilon - \mathbf{v}) \ \text{ in } \Omega. \end{array} \right. \tag{11}$$

We notice that any solution $\mathbf{v}_\phi^\varepsilon$ to (11) writes exactly as a gradient and necessarily verifies $\mathbf{rot}\,\mathbf{v}_\phi^\varepsilon = 0$. Indeed, this method can be viewed as a suitable approximate method to find, at the limit process when $\varepsilon \to 0$, the unique solution $\mathbf{v}_\phi$ in $\mathbf{H}_{div}(\Omega)$ of the exact orthogonal projection problem of $\mathbf{v}$ onto $\mathbf{G}$:

$$\mathrm{div}\,\mathbf{v}_\phi = \mathrm{div}\,\mathbf{v} \quad \text{and} \quad \mathbf{rot}\,\mathbf{v}_\phi = 0 \ \text{ in } \Omega \quad \text{with} \quad \mathbf{v}_\phi \cdot \mathbf{n}_{|\Gamma} = \mathbf{v}\cdot\mathbf{n} \ \text{ on } \Gamma. \tag{12}$$

The problem (VPP_n) is well-posed in $\mathbf{H}_{div}(\Omega)$ as stated in Theorem 3.3 below; see proof in [7].

Theorem 3.3 (Analysis of the vector penalty-projection (VPP$_n$).)**.** *For any $\mathbf{v} \in \mathbf{H}_{div}(\Omega)$ and all $\varepsilon > 0$, there exists a unique solution $\mathbf{v}_\phi^\varepsilon$ in $H_{div}(\Omega)$ to the vector penalty-projection (11). Moreover, $\mathbf{v}_\phi^\varepsilon$ is curl-free: $\mathbf{rot}\,\mathbf{v}_\phi^\varepsilon = 0$, $\mathbf{v}_\phi^\varepsilon = \mathbf{grad}\,\phi^\varepsilon \in \mathbf{G}$ and $div\,(\mathbf{v}_\phi^\varepsilon - \mathbf{v}) \in H^1(\Omega) \cap L_0^2(\Omega)$ for all $\varepsilon > 0$. Then, we can choose $\phi^\varepsilon \in H^1(\Omega) \cap L_0^2(\Omega)$ such that $div\,(\mathbf{v}_\phi^\varepsilon - \mathbf{v}) = \varepsilon\,\phi^\varepsilon$.*

Besides, we have the following error estimates:

$$\|\mathbf{v}_\phi - \mathbf{v}_\phi^\varepsilon\|_1 + \|\phi - \phi^\varepsilon\|_2 + \|div\,(\mathbf{v} - \mathbf{v}_\phi^\varepsilon)\|_1 \le c(\Omega)\,\|\mathbf{v}\|_0\,\varepsilon, \qquad \textit{for all } \ \varepsilon > 0. \tag{13}$$

A discrete scalar potential ϕ^ε can be also reconstructed directly from its gradient $\mathbf{grad}\,\phi^\varepsilon = \mathbf{v}_\phi^\varepsilon$ with a fast algorithm performing a circulation along a suitable path joining the potential nodes in the unstructured mesh, as presented in [4]. Let us also notice that (11) corresponds to the vector correction step performed at each time step in the proposed (VPP$_\varepsilon$) method [4, 6] to solve the Navier-Stokes equations, whereas $\mathbf{v} = -\widetilde{\mathbf{v}}$ is calculated by a prediction step which does not take into account the divergence-free constraint.

Remark 2 (Approximation of the harmonic vector $\mathbf{v}_h = \mathbf{grad}\,\phi_h$.)*. The field $\mathbf{v}_h^\varepsilon = \mathbf{grad}\,\phi_h^\varepsilon$ and ϕ_h^ε can be calculated by: $\mathbf{v}_h^\varepsilon = \mathbf{v}_\phi^\varepsilon - \mathbf{v}_0^\varepsilon$ which is exactly a gradient and $\phi_h^\varepsilon = \phi^\varepsilon - \phi_0^\varepsilon$ (up to an additive constant). Doing this, we have $\|div\,\mathbf{v}_h^\varepsilon\|_0 = \mathcal{O}(\varepsilon)$ whereas $\mathbf{rot}\,\mathbf{v}_h^\varepsilon$ is exactly zero whatever the penalty parameter ε.*

Remark 3 (Order of approximation.). *The (VPP) method yields approximations of order $\mathcal{O}(\varepsilon)$ of $\mathbf{v}_\phi$ and ϕ with $\mathbf{v}^\varepsilon_\phi = \mathbf{grad}\,\phi^\varepsilon$ in $\mathbf{G}$. However, the divergence of $(\mathbf{v}-\mathbf{v}^\varepsilon_\phi)$ is not exactly zero, only $\mathcal{O}(\varepsilon)$, which prevents us from representing it exactly as a curl and thus from directly computing an approximation of the vector potential $\boldsymbol{\psi}$. This is performed with the (RPP) method presented in Section 2. Therefore, the two approximate components $\mathbf{v}^\varepsilon_\psi = \mathbf{rot}\,\boldsymbol{\psi}^\varepsilon \in \mathbf{H}$ and $\mathbf{v}^\varepsilon_\phi = \mathbf{grad}\,\phi^\varepsilon \in \mathbf{G}$ are always rigourously orthogonal in $\mathbf{L}^2(\Omega)$, whatever the penalty parameter ε.*

4. Numerical results with Discrete Operator Calculus methods

The discretization method with Discrete Operator Calculus is an extension of the MAC (Marker And Cell) method with staggered grids [19] to unstructured meshes. The method is similar to Discrete Exterior Calculus (DEC) based on differential geometry [23]. The scheme is based on a node-center approach avoiding interpolations, where the scalar or vector components unknowns are distributed on nodes, faces and edges of the mesh stencils; see more details in [10, 26, 7]. The primal and dual meshes enable to express gradient, divergence, curl operators as well as Green, Gauss and Stokes theorems in such a way that the 2-D or 3-D discrete operators satisfy, as in the continuum case, the following properties whatever the mesh step h in Ω: $\mathrm{div}_h(\mathbf{rot}_h\boldsymbol{\psi}) = 0$ and $\mathbf{rot}_h(\mathbf{grad}_h\phi) = 0$ up to machine precision. Indeed, this is verified by our discretization as shown in Figure 1 and it is in agreement with the calculation given in [26, Appendix C].

The discretization is shown to locally and globally conserve up to machine precision, mass, kinetic energy and vorticity in the absence of viscosity; see [9]. We have experimented that the spatial accuracy is of second-order on a structured or unstructured mesh both in 2-D or 3-D, including highly irregular meshes, as for MAC grids in [19].

We consider below the vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ given in the square domain $\Omega =]-0.5, 0.5[\times]-0.5, 0.5[$:

$$\mathbf{v} = (\sin(\pi(x+y)) + 1)\ \mathbf{e}_x + (\sin(\pi(y-x)) + 0.5)\ \mathbf{e}_y.$$

It is provided from the Helmholtz-Hodge orthogonal decomposition $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_h + \mathbf{v}_\psi = \mathbf{v}_\phi + \mathbf{v}_\psi$ with $\mathbf{v}_\phi = \mathbf{v}_0 + \mathbf{v}_h$ and the following curl-free or divergence-free components:

$$\begin{aligned}
\mathbf{v}_0 = \mathbf{grad}\,\phi_0 = \sin(\pi x)\ \cos(\pi y)\,\mathbf{e}_x + \cos(\pi x)\ \sin(\pi y)\,\mathbf{e}_y \qquad &\text{with} \qquad \phi_0 = -\frac{1}{\pi}\ \cos(\pi x)\ \cos(\pi y)\\
\mathbf{v}_h = \mathbf{grad}\,\phi_h = 1\,\mathbf{e}_x + 0.5\,\mathbf{e}_y \qquad &\text{with} \qquad \phi_h = x + 0.5\,y\\
\mathbf{v}_\psi = \mathbf{rot}\,\boldsymbol{\psi} = \cos(\pi x)\ \sin(\pi y)\,\mathbf{e}_x - \sin(\pi x)\ \cos(\pi y)\,\mathbf{e}_y \qquad &\text{with} \qquad \boldsymbol{\psi}\cdot\mathbf{e}_z = -\frac{1}{\pi}\ \cos(\pi x)\ \cos(\pi y).
\end{aligned}$$

These components and related scalar or vector potentials are computed with the (RPP) and (VPP) discrete problems on a 64×64 uniform mesh (for the mesh step $h = 1/64$) with the penalty parameter $\varepsilon = 10^{-14}$. The different fields are represented in Figure 3. We observe that the errors on all these fields vary as $\mathcal{O}(h^2)$ in the $\mathbf{L}^2$-norms, like in [6], since the penalization error in $\mathcal{O}(\varepsilon)$ is always negligible with respect to the discretization error. Moreover, the orthogonality properties are verified up to machine precision.

Another key point is the very fast convergence of the preconditioned conjugate gradient solvers: typically only two iterations are necessary to reach the machine precision whatever the mesh size, as shown in Figure 2, which is incredibly effective. This is in perfect agreement with [5, Theorem 1.1 and Corollary 1.3], the very good conditioning property with adapted right-hand sides being addressed in [5, Corollary 1.2], and the results in [3, 6] obtained for (VPP) discrete problems. These theoretical results can be applied as well for (RPP) discrete problems by using the following equality which holds for any vector field $\mathbf{v}$ in 2-D or 3-D:

$$-\boldsymbol{\Delta}\mathbf{v} = \mathbf{rot}\ (\mathbf{rot}\,\mathbf{v}) - \mathbf{grad}\ (\mathrm{div}\,\mathbf{v})\,.$$

Acknowledgements

We thank the anonymous referee for his (or her) careful review of our manuscript which improved the revised version.

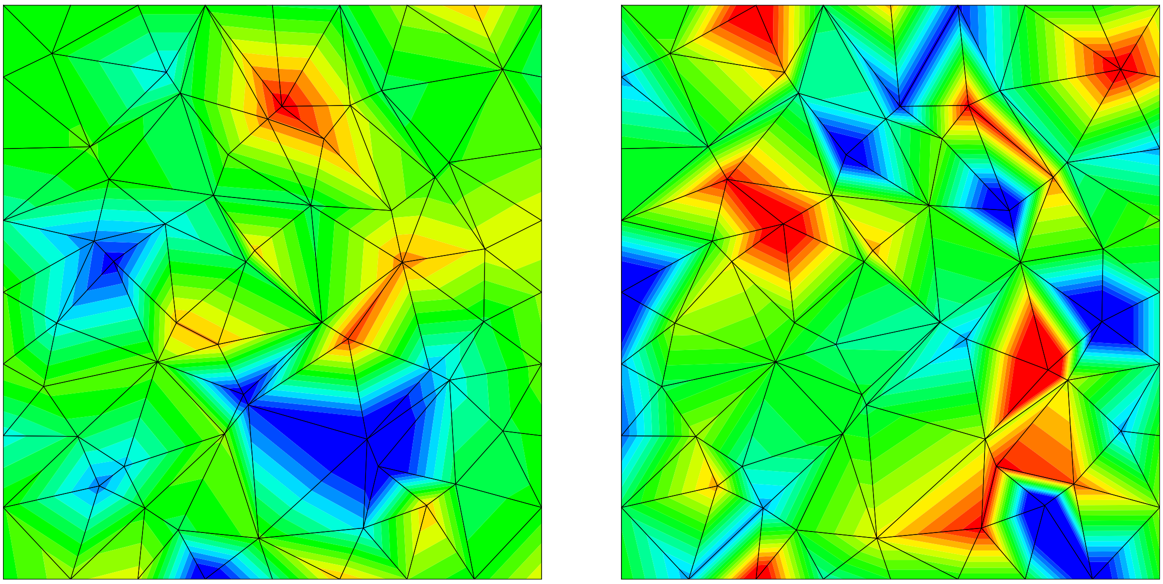

Figure 1: Discrete Exterior Calculus identities on a random Delaunay mesh for a typical analytic scalar field ϕ or vector field $\boldsymbol{\psi}$. Left: $\mathbf{rot}_h(\mathbf{grad}_h\phi) = \pm 1.7\,10^{-15}$ in Ω – Right: $\mathrm{div}_h(\mathbf{rot}_h\boldsymbol{\psi}) = \pm 1.4\,10^{-14}$ in Ω.

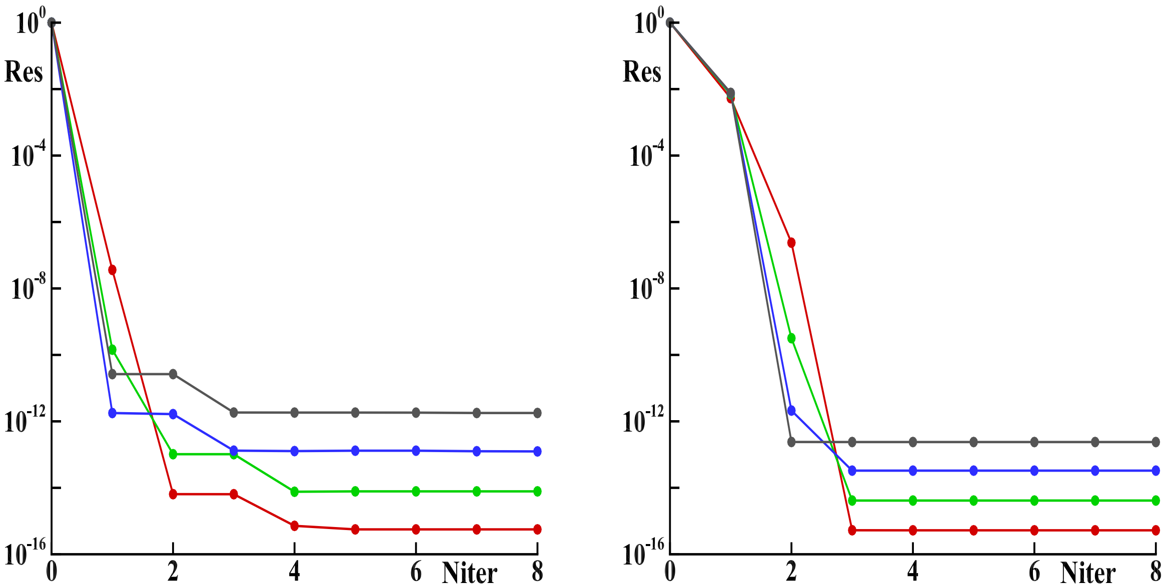

Figure 2: Convergence of BiCGstab2-ILU(0) for (RPP) or (VPP) problems with $\varepsilon = 10^{-14}$: normalized residual (by initial residual) versus number of iterations for different mesh sizes 32×32 (red), 128×128 (green), 512×512 (blue) and 2048×2048 (black); solvers started with zero initial guess – Left: Rotational Penalty-Projection (RPP). Right: Vector Penalty-Projection (VPP).

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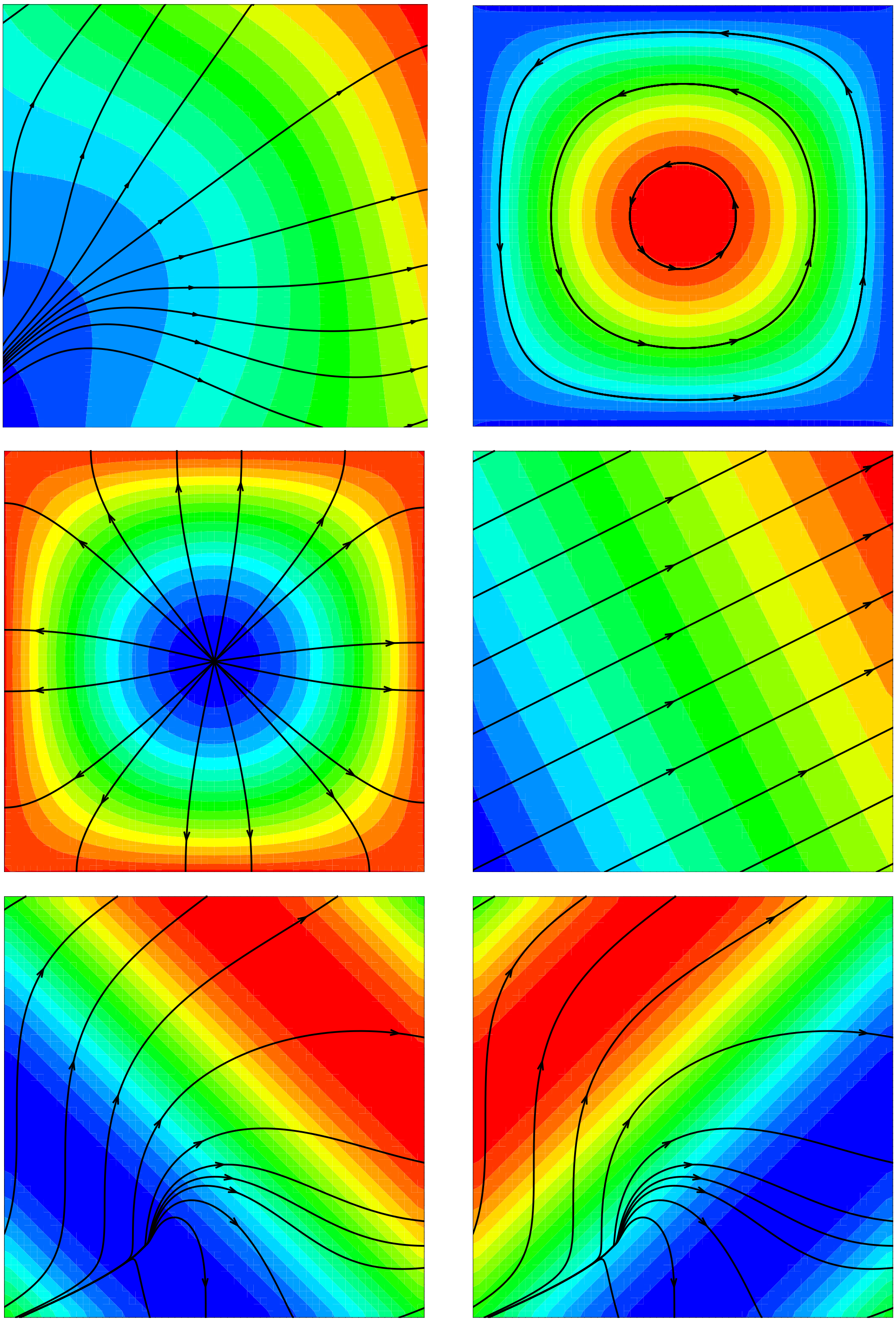

Figure 3: DHHD extracted fields with (RPP) and (VPP) methods for $\varepsilon = 10^{-14}$ and mesh size $= 64 \times 64$ – Top Left: potential ϕ. Top Right: potential $\boldsymbol{\psi} \cdot \mathbf{e}_z$. Middle Left: potential ϕ_0. Middle Right: harmonic potential ϕ_h. Bottom Left: horizontal component of the reconstructed field $\mathbf{v}$. Bottom Right: vertical component of the reconstructed field $\mathbf{v}$.